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Existence Theorems of Simultaneous Equilibrium Problems and Generalized Vector Quasi-Saddle Points

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Abstract. In this paper, we establish the existence theorems of simultaneous equilibrium problems. As consequences of our results, we establish the existence theorem of simultaneous mathematical programs and equilibrium problems and the existence theorems of generalized vector quasi-saddle point problems.

Key words: closed multivalued map, $C(x)$-convex, C-diagonally quasi-convex, Fréchet derivative, generalized vector quasi-saddle point

1. Introduction

Let E be a topological vector space, $X \subset E$ be a nonempty subset, $f: X \times X \to \mathbb{R}$ be a bifunction with $f(x,x)=0$ for all $x \in E$, the scalar equilibrium problem (EP) is to fined $\bar{x} \in X$ such that

$$f(\bar{x}, y) \geqslant 0 \quad \text{for all } y \in X.$$

The equilibrium problem encompasses, as special cases, many important problems including optimization problems, variational inequalities problems, Nash equilibrium problems, fixed point problems and complementary problems (see Blum and Oettli, 1994). This type of problem is extensively investigated and generalized by many authors (see Bianchi and Schaible, 1996; Ansari et al., 1997; Bianchi et al., 1997; Lin and Park, 1998; Ansari and Yao, 1999; Ansari, 2000; Lin and Yu, 2001; Lin et al., 2002; Hou et al., 2003; Lin et al., 2003 and references there in).

Let Z be a Hausdorff topological vector space (in short t.v.s.) and let X and Y be nonempty subsets of two Hausdorff t.v.s., respectively. Let $S: X \multimap X$, $T: X \multimap Y$, $C: X \multimap Z$, $f: X \times Y \times X \multimap Z$ and $g: X \times Y \times Y \multimap Z$ be multivalued maps with nonempty values.

In this paper, we study the following classes of simultaneous generalized vector quasi-equilibrium problems:

(I) Find $(\bar{x}, \bar{y}) \in X \times Y$, $\bar{x} \in S(\bar{x})$ and $\bar{y} \in T(\bar{x})$ such that

$$f(\bar{x}, \bar{y}, u) \subseteq C(\bar{x}) \quad \text{for all } u \in S(\bar{x}) \quad \text{and}$$
$$g(\bar{x}, \bar{y}, v) \subseteq C(\bar{x}) \quad \text{for all } v \in T(\bar{x}).$$

(II) Find $(\bar{x}, \bar{y}) \in X \times Y$, $\bar{x} \in S(\bar{x})$ and $\bar{y} \in T(\bar{x})$ such that

$$f(\bar{x}, \bar{y}, u) \cap C(\bar{x}) \neq \emptyset \quad \text{for all } u \in S(\bar{x}) \quad \text{and}$$
$$g(\bar{x}, \bar{y}, v) \cap C(\bar{x}) \neq \emptyset \quad \text{for all } v \in T(\bar{x}).$$

(III) Find $(\bar{x}, \bar{y}) \in X \times Y$, $\bar{x} \in S(\bar{x})$ and $\bar{y} \in T(\bar{x})$ such that

$$f(\bar{x}, \bar{y}, u) \cap (-\operatorname{int} C(\bar{x})) = \emptyset \quad \text{for all } u \in S(\bar{x}) \quad \text{and}$$
$$g(\bar{x}, \bar{y}, v) \cap (-\operatorname{int} C(\bar{x})) = \emptyset \quad \text{for all } v \in T(\bar{x}).$$

(IV) Find $(\bar{x}, \bar{y}) \in X \times Y$, $\bar{x} \in S(\bar{x})$ and $\bar{y} \in T(\bar{x})$ such that

$$f(\bar{x}, \bar{y}, u) \not\subseteq -\operatorname{int} C(\bar{x}) \quad \text{for all } u \in S(\bar{x}) \quad \text{and}$$
$$g(\bar{x}, \bar{y}, v) \not\subseteq -\operatorname{int} C(\bar{x}) \quad \text{for all} \quad v \in T(\bar{x}).$$

If $g = 0$, then the above four kinds of simultaneous generalized vector quasi-equilibrium problems are reduced to the vector quasi-equilibrium problems studied by Hou et al. (2003).

(a) If $g = 0$, $Y = X$ and $T(x) = X$, $S(x) = X$ for all $x \in X$, and $F(x, u) = f(x, y, u)$ for all $(x, y, u) \in X \times X \times X$, then equilibrium problem IV is reduced to the problem, which was studied in Ansari et al. (1997) and Lin et al. (2003). If we assume further that $C(x) = Z \setminus D(x)$, then equilibrium problem II is reduced to the problem, which was studied in Lin et al. (2002).
(b) If $g = 0$ and $C(x) = C$ for all $x \in X$, where C is a cone in Z. Then equilibrium problem, III is reduced to the problem, which was studied in Lin and Yu (2001).
(c) If $g = 0$, $Y = X$ and $T(x) = X$, $S(x) = X$, $C(x) = C$, $F(x, u) = f(x, y, u)$ for all $x, y, u \in X$. These equilibrium problems were studied in Ansari (2000).
(d) If $g = 0$, $C(x) = C$ for all $x \in X$, and $f: X \times Y \times X \to Z$. Then equilibrium problems III and IV were reduced to the problem, which was considered in Lin and Yu (2001).
(e) If $g = 0$, $Z = \mathbb{R}$ and $C(x) = \mathbb{R}^+$ for all $x \in X$ and $f: X \times Y \times X \longrightarrow \mathbb{R}$, then problems I–IV are the same. This problem was considered in Lin and Park (1998).

The first part of this paper is to study the existence theorems of four kinds of simultaneous generalized vector quasi-equilibrium problems. Our results extend and improve all the results in Hou et al. (2003), and our results do not assume any kind of monotonicty assumption. As applications of our results, we study the existence theorem of the following simultaneous mathematical program and equilibrium problem.

Find $(\bar{x}, \bar{y}) \in X \times Y$, $\bar{x} \in S(\bar{x})$ and $\bar{y} \in T(\bar{x})$ such that

$$f(u) - f(\bar{x}) \in C(\bar{x}) \quad \text{for all } u \in S(\bar{x})$$

and

$$g(\bar{x}, \bar{y}, v) \in C(\bar{x}) \quad \text{for all } v \in T(\bar{x}),$$

where $f: X \multimap Z$ and $C: X \multimap Z$, $g: X \times Y \times Y \to Z$.

We also apply the existence theorems of the simultaneous generalized vector quasi-equilibrium problem to study the generalized vector quasi-saddle point problems.

Find $(\bar{x}, \bar{y}) \in X \times Y$, $\bar{x} \in S(\bar{x})$ and $\bar{y} \in T(\bar{x})$ such that

$$f(x, \bar{y}) - f(\bar{x}, \bar{y}) \in C(\bar{x}) \quad \text{for all } x \in S(\bar{x})$$

and

$$f(\bar{x}, \bar{y}) - f(\bar{x}, y) \in C(\bar{x}) \quad \text{for all } y \in T(\bar{x}).$$

Recently, Kazm et al. (2001) study the vector saddle point with constant cone, Kimura and Tanaka (2003) study the saddle point, when $S(x) = X$ and $T(x) = Y$, and $C(x)$ is replaced by $\operatorname{int} C(x)$. Our results and our approach are quite different from the existence results in the literature (e.g., Luc and Vargas, 1992; Tanaka, 1994, 1998, 1999; Shi and Ling, 1995; Kazmi et al., 2001; Kimura and Tanaka, 2003).

2. Preliminaries

Let X be a nonempty subset of a topological space E. We denote by 2^X the family of all subsets of the set X, by $\langle X \rangle$ the class of all finite subset of X, by $\bar{X}$ the closure of X, and by $\operatorname{int} X$ the interior of X.

For nonempty sets X and Y, a multivalued map $F: X \multimap Y$ is a function from X into 2^Y. Let X and Y be topological spaces and $T: X \multimap Y$ be a multivalued map.

(1) T is upper semicontinuous (in short u.s.c.) (resp. lower semicontinuous, in short l.s.c) at $x \in X$ if for every open set V containing $T(x)$ (resp.

$T(x) \cap V = \emptyset$), there is an open set U containing x such that $T(u) \subseteq V$(resp. $T(U) \cap V = \emptyset$) for all $u \in U$; T is u.s.c. on X if T is u.s.c. (resp. l.s.c) at every point of X.

(2) T is continuous at x if T is both u.s.c. and l.s.c. at x.

(3) T is closed if $G_r T = \{(x, y) \in X \times Y : y \in T(x)\}$ is closed in $X \times Y$.

(4) T is compact if there exists a compact set K such that $T(X) \subseteq K$.

Throughout this paper, all topological spaces are assumed to be Hausdorff.

The following definitions and theorems are need in this paper.

DEFINITION 2.1. Let X be a convex subset of a t.v.s. and Z be a t.v.s. Let $f : X \times X \multimap Z$, $g : X \multimap Z$, and $C : X \multimap Z$ be multivalued maps. Given any $\Lambda = \{x_1, x_2, \dots, x_n\} \in \langle X \rangle$ and any $x \in \mathrm{co}\{x_1, x_2, \dots, x_n\}$.

(1) f is said to be strong type I C-diagonally quasi-convex (SIC-DQC, in short) (Hou et al., 2003) in the second argument if for some $x_i \in \Lambda$.

$$f(x, x_i) \subseteq C(x);$$

(2) f is said to be strong type II C-diagonally quasi-convex (SIIC-DQC, in short) (Hou et al., 2003) in the second argument if for some $x_i \in \Lambda$.

$$f(x, x_i) \cap C(x) \neq \emptyset;$$

(3) f is said to be weak type I C-diagonally quasi-convex (WIC-DQC, in short) (Hou et al., 2003) in the second argument if for some $x_i \in \Lambda$.

$$f(x, x_i) \cap (-\mathrm{int}\, C(x)) = \emptyset;$$

(4) f is said to be weak type II C-diagonally quasi-convex (WIIC-DQC, in short) (Hou et al., 2003) in the second argument if for some $x_i \in \Lambda$.

$$f(x, x_i) \not\subseteq -\mathrm{int}\, C(x);$$

(5) g is said to be convex (resp. concave) if for any $x_1, x_2 \in X$, $\lambda \in [0, 1]$,

$$g(\lambda x_1 + (1-\lambda)x_2) \subseteq \lambda g(x_1) + (1-\lambda) g(x_2),$$
$$(\text{resp. } \lambda g(x_1) + (1-\lambda) g(x_2) \subseteq g(\lambda x_1 + (1-\lambda)x_2));$$

(6) g is said to be C_x-quasiconcave like if for any $x, y \in X$, $\lambda \in [0, 1]$, either

$$g(\lambda x + (1-\lambda)y) \subseteq g(x) + C(x)$$
$$\text{or } g(\lambda x + (1-\lambda)y) \subseteq g(y) + C(x);$$

(7) g is said to be C_x-quasiconconvex if for any $x, y \in X$, $\lambda \in [0, 1]$, either

$$g(x) \subseteq g(\lambda x + (1-\lambda)y) + C(x)$$
$$\text{or } g(y) \subseteq g(\lambda x + (1-\lambda)y) + C(x);$$

(8) g is said to be $C(x)$ convex if $\alpha g(x_1) + (1-\alpha)g(x_2) - g(\alpha x_1 + (1-\alpha)x_2) \subset C(\alpha x_1 + (1-\alpha)x_2)$ for all x_1, $x_2 \in X$ and $\alpha \in [0, 1]$.

DEFINITION 2.2. Let X and Y be Banach spaces, $f: X \to Y$, f is said to be Fréchet differentiable at $x_0 \in X$, if there exists a $Df(x_0) \in L(X, Y)$ such that

$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - \langle Df(x_0), x - x_0 \rangle\|}{\|x - x_0\|} = 0,$$

where $L(X, Y) = \{T | T: X \to Y$ is a continuous linear operator$\}$. $Df(x_0)$ is said to be the Fréchet derivative of f at x_0, f is said to be Fréchet differentiable on X if f is Fréchet differentiable at each point of X.

THEOREM 2.1. (Aubin and Cellina, 1994). *Let X and Y be topological spaces, $T: X \multimap Y$ be a multivalued map.*

(1) *If T is an u.s.c. multivalued map with closed values, then T is closed.*
(2) *If T is closed and Y is compact, then T is an u.s.c. multivalued map.*
(3) *If X is compact and $T: X \multimap Y$ is an u.s.c. multivalued map with compact values, then $T(X)$ is compact.*

THEOREM 2.2. (Tan, 1995). *Let T be a multivalued map of a topological spaces X into a topological spaces Y. Then T is l.s.c. at $x \in X$ if and only for any $y \in T(x)$ and for any net $\{x_\alpha\}$ in X converging to x, there is a net $\{y_\alpha\}$ such that $y_\alpha \in T(x_\alpha)$ for every α and y_α converging to y.*

THEOREM 2.3. (Kim and Tan, 2001). *Let X and Y be nonempty compact convex metrizable subsets of locally convex t.v.s E and H, respectively. A: $X \multimap X$, $F: X \multimap Y$ and $P: X \times Y \multimap X$ be multivalued maps satisfying the following conditions:*

(1) *For each $x \in X$, $A(x)$ is a nonempty convex subset of X;*
(2) *$clA: X \multimap X$ is u.s.c.;*
(3) *$F: X \multimap X$ is u.s.c. and $F(x)$ is a nonempty closed convex subset of X for all $x \in X$;*
(4) *For all $(x, y) \in X \times Y$, $x \notin coP(x, y)$;*
(5) *For all $y \in Y$, $P^{-}(y)$ and $A^{-}(y)$ is open in $X \times Y$.*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in \mathrm{cl}A(\bar{x})$, $\bar{y} \in F(\bar{x})$ and $A(\bar{x}) \cap P(\bar{x}, \bar{y}) = \emptyset$.

THEOREM 2.4. (Ding and Tarafdar, 2000). *Let W and Z be t.v.s. Suppose that $L(W, Z)$ is equipped with σ-topology. Then the bilinear mapping $<\cdot, \cdot>$: $L(W, Z) \times Z$ is continuous on $L(W, Z) \times W$.*

THEOREM 2.5. (Swartz, 1992). *Let X be a complete metrizable locally convex t.v.s. If $K \subset X$ is compact, then $\overline{co}K$ is compact.*

PROPOSITION 2.6. *Let K be a convex space, Z a t.v.s., $F: K \times K \multimap Z$ and $C: K \multimap Z$ be multivalued maps such that $C(x)$ is a convex cone. Then F is C_x − quasiconvex if and only if for any $x \in K$, $y_i \in K$, $t_i > 0$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} t_i = 1$, there exists $1 \leqslant j \leqslant n$ such that*

$$F(x, y_j) \subseteq F\left(x, \sum_{i=1}^{n} t_i y_i\right) + C(x). \tag{1}$$

Proof. The sufficiency is obvious. Suppose that F is C_x quasiconvex. We can prove (1) immediately by induction.

THEOREM 2.7. (Fan, 1961). *Let E be a t.v.s., $X \subseteq E$ be an arbitrary set, and $G: X \multimap E$ a KKM map. If $G(x)$ is closed for all $x \in X$ and $G(x_0)$ is compact for some $x_0 \in X$. Then $\bigcap\{G(x): x \in X\} \neq \emptyset$.*

3. Simultaneous equilibrium problems

Throughout this section unless otherwise specify,we assume that Z is a real t.v.s. and X and Y are two nonempty compact convex metrizable sets in two locally convex t.v.s., respectively, $f: X \times Y \times X \multimap Z$, $g: X \times Y \times Y \multimap Z$, $C: X \multimap Z$, $T: X \multimap Y$ and $S: X \multimap X$ are multivalued maps with nonempty values.

THEOREM 3.1. *Suppose that*

(i) *$S: X \multimap X$ is a multivalued map with nonempty convex values and $S^-(y)$ is open for all $y \in X$ and $\mathrm{cl}S: X \multimap X$ is u.s.c.;*
(ii) *$T: X \multimap Y$ is continuous with nonempty closed convex values;*
(iii) *C is closed and for each $x \in X$, $C(x)$ is a nonempty convex cone;*
(iv) *g is l.s.c. $g(x, y, y) \subset C(x)$ for all $(x, y) \in X \times Y$;*
(v) *(a) for any $u \in X$, the set $\{(x, y) \in X \times Y: f(x, y, u) \not\subseteq C(x)\}$ is open in $X \times Y$;*

(b) for all $y \in Y$, the function $f(\cdot, y, \cdot)$ is SIC-DQC in the third argument.

(vi) *for each fixed $(x, v) \in X \times Y$, $y \multimap g(x, y, v)$ is C_x-quasiconcave like, and for each $(x, y) \in X \times Y$, $v \multimap g(x, y, v)$ is C_x-quasiconvex.*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$, $\bar{x} \in cl S(\bar{x})$, $\bar{y} \in T(\bar{x})$ such that

$$f(\bar{x}, \bar{y}, u) \subseteq C(\bar{x}) \quad \text{for all } u \in S(\bar{x})$$

and

$$g(\bar{x}, \bar{y}, v) \subseteq C(\bar{x}) \quad \text{for all } v \in T(\bar{x}).$$

Proof. Let $H: X \multimap Y$ and $P: X \times Y \multimap X$ be defined by

$$H(x) = \{y \in T(x): g(x, y, v) \subseteq C(x) \quad \text{for all } v \in T(x)\}$$

and $P(x, y) = \{u \in X: f(x, y, u) \nsubseteq C(x)\}$ for $(x, y) \in X \times Y$. We want to show that $H(x)$ is nonempty for all $x \in X$. For each $x \in X$, let $G_x: T(x) \multimap T(x)$ be defined by $G_x(v) = \{y \in T(x): g(x, y, v) \subseteq C(x)\}$. We show that G_x is a KKM map. Suppose that there exists a finite set $\{v_1, v_2, \ldots, v_n\}$ in $T(x)$ such that $co\{v_1, v_2, \ldots, v_n\} \nsubseteq \bigcup_{i=1}^{n} G_x(v_i)$. Then there exists $\overline{y} \in co\{v_1, v_2, \ldots, v_n\}$ such that $\overline{y} \notin G_x(v_i)$ for all $i = 1, 2, \ldots, n$.

Since $T(x)$ is convex for all $x \in X$ and $\{v_1, v_2, \ldots, v_n\} \subseteq T(x)$, $\overline{y} \in T(x)$. But $\overline{y} \notin G_x(v_i)$ for all $i = 1, 2, \ldots, n$. Therefore, $g(x, \overline{y}, v_i) \nsubseteq C(x)$ for all $i = 1, 2, \ldots, n$. By (vi), (iv) and Proposition 2.6 that there exists $1 \leqslant j \leqslant n$ such that $g(x, \overline{y}, v_j) \subseteq g(x, \overline{y}, \overline{y}) + C(x) \subseteq C(x)$. This leads to a contradiction. Therefore, $G_x: T(x) \multimap T(x)$ is a KKM map for each fixed $x \in X$.

For each fixed $x \in X$ and $v \in Y$, $G_x(v)$ is closed. Indeed, let $y \in \overline{G_x(v)}$, then there exists a net $\{y_\alpha\}$ in $G_x(v)$ such that $y_\alpha \to y$. Therefore, $y_\alpha \in T(x)$ and $g(x, y_\alpha, v) \subset C(x)$. By (ii) and Theorem 2.1 that T is closed, and $y \in T(x)$. Let $z \in g(x, y, v)$. It follows from (iv) and Theorem 2.2 that there exists a net $\{z_\alpha\}$ in $g(x, y_\alpha, v)$ such that $z_\alpha \to z$. By (iii), $C: X \multimap Z$ is closed, $C(x)$ is a closed set for each $x \in X$. Since $z_\alpha \in g(x, y_\alpha, v) \subseteq C(x)$, $z \in C(x)$ and $g(x, y, v) \subseteq C(x)$. This shows that $G_x(v)$ is closed. Since $G_x(v) \subseteq T(x)$ and $T(x)$ is compact for each $x \in X$, $G_x(v)$ is compact for each $x \in X$ and $v \in Y$. By Theorem 2.7 that $\cap_{v \in T(x)} G_x(v) \neq \emptyset$. Let $y \in \cap_{v \in T(x)} G_x(v)$. Then $y \in G_x(v)$ for all $v \in T(x)$. Therefore for all $x \in X$, $H(x) \neq \emptyset$.

H is closed. Indeed, if $(x, y) \in \overline{G_r H}$, then there exists a net $\{(x_\alpha, y_\alpha)\}$ in $G_r H$ such that $(x_\alpha, y_\alpha) \to (x, y)$. Therefore, $y_\alpha \in T(x_\alpha)$ and $g(x_\alpha, y_\alpha, v) \subseteq C(x_\alpha)$ for all $v \in T(x_\alpha)$. Let $v' \in T(x)$ and $w \in g(x, y, v')$. Since T and g are l.s.c. and $x_\alpha \to x$, there exist nets $\{v_\alpha\}$ and $\{w_\alpha\}$ such that $v_\alpha \to v'$, $v_\alpha \in T(x_\alpha)$ for all α, $w_\alpha \to w$ and $w_\alpha \in g(x_\alpha, y_\alpha, v_\alpha) \subset C(x_\alpha)$ for all α. Since C is closed,

$w \in C(x)$ and $g(x, y, v') \subseteq C(x)$ for all $v' \in T(x)$. Since T is closed, $y \in T(x)$ and H is closed. Since Y is compact, it follows from Theorem 2.1 that $H: X \multimap Y$ is u.s.c. with closed values. Let $y_1, y_2 \in H(x)$ and $0 \leqslant \lambda \leqslant 1$, then $y_1, y_2 \in T(x)$ and $g(x, y_1, v) \subseteq C(x)$ and $g(x, y_2, v) \subseteq C(x)$ for any $v \in T(x)$. Let $y_\lambda = \lambda y_1 + (1-\lambda) y_2$. By (vi), for any $v \in T(x)$, either

$$g(x, y_\lambda, v) \subseteq g(x, y_1, v) + C(x) \subseteq C(x) + C(x) \subseteq C(x)$$

or

$$g(x, y_\lambda, v) \subseteq g(x, y_2, v) + C(x) \subseteq C(x) + C(x) \subseteq C(x).$$

Since $T(x)$ is convex for each $x \in X$, $y_\lambda \in T(x)$. Therefore, $y_\lambda \in H(x)$ and $H(x)$ is convex. By (v)(b), it is easy to see that for each $(x, y) \in X \times Y$, $x \notin \mathrm{co}P(x, y)$. By (v)($a$) $P^-(u)$ is open for all $u \in X$. It follows from Theorem 2.3 that there exists $(\bar{x}, \bar{y})$ in $X \times Y$ with $\bar{x} \in clS(\bar{x})$ and $\bar{y} \in H(\bar{x})$ such that $S(\bar{x}) \cap P(\bar{x}, \bar{y}) = \emptyset$. Therefore, $\bar{y} \in T(\bar{x})$, $\bar{x} \in cl\ S(\bar{x})$ $f(\bar{x}, \bar{y}, u) \subseteq C(\bar{x})$ for all $u \in S(\bar{x})$, and $g(\bar{x}, \bar{y}, v) \subseteq C(\bar{x})$ for all $v \in T(\bar{x})$.

REMARK. If $g = 0$, then Theorem 3.1 is reduced to Theorem 3.5 in Hou et al., 2003

COROLLARY 3.1. *In Theorem* 3.1, *if condition* (v) *(a) is replaced by (a′) for each fixed* $u \in X$, $(x, y) \to f(x, y, u)$ *is l.s.c. Then the conclusion of Theorem 3.1 holds.*

Proof. Following the same argument as in Theorem 3.1, we can show that for each $u \in X$, $A(u) = \{(x, y) \in X \times Y : f(x, y, u) \subseteq C(x)\}$ is closed.

THEOREM 3.2. *Let* X, Y, Z, S *and* T *be the same as in Theorem* 3.1. *Suppose that*

(i) *for each* $x \in X$, $C(x)$ *is a proper convex cone and* $\mathrm{int}\, C(x) \neq \emptyset$;
(ii) *for each* $(x, v) \in X \times Y$, $y \to g(x, y, v)$ *is* C_x*-quasiconcave and* g *is an u.s.c. multivalued map with compact values; for each* $(x, y) \in X \times Y$, $v \multimap g(x, y, v)$ *is* C_x*-quasiconvex and* $g(x, y, y) \subseteq C(x)$ *for all* $x \in X$ *and* $y \in Y$;
(iii) $W: X \multimap Z$ *defined by* $W(x) = Z \backslash (-\mathrm{int}\, C(x))$ *is u.s.c.;*
(iv) *(a) for any* $u \in X$, *the set* $\{(x, y) \in X \times Y : f(x, y, u) \subseteq -\mathrm{int}\, C(x)\}$ *is open in* $X \times Y$;
(b) for any $y \in Y$, *the multivalued map* $f(\cdot, y, \cdot)$ *is WIIC-DQC in the third argument.*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$, $\bar{x} \in clS(\bar{x})$ and $\bar{y} \in T(\bar{x})$ such that

$$f(\bar{x}, \bar{y}, u) \nsubseteq -\mathrm{int}\, C(\bar{x}) \quad \text{for all } u \in S(\bar{x})$$

and

$$g(\bar{x}, \bar{y}, v) \nsubseteq -\mathrm{int}\, C(\bar{x}) \quad \text{for all } v \in T(\bar{x}).$$

Proof. Let $H: X \multimap Y$ and $P: X \times Y \multimap X$ be defined by $H(x) = \{y \in T(x): g(x, y, v) \nsubseteq -\mathrm{int}\, C(x)$ for all $v \in T(x)\}$, and $P(x, y) = \{u \in X: f(x, y, u) \subseteq -\mathrm{int}\, C(x)\}$ for $(x, y) \in X \times Y$. Let $y_1, y_2 \in H(x)$ and $\lambda \in [0, 1]$. Then $y_1, y_2 \in T(x)$ and $g(x, y_1, v) \nsubseteq -\mathrm{int}\, C(x)$ and $g(x, y_2, v) \nsubseteq -\mathrm{int}\, C(x)$ for all $v \in T(x)$. Let $y_\lambda = \lambda y_1 + (1-\lambda) y_2$. Since $T(x)$ is convex for all $x \in X$, $y_\lambda \in T(x)$. We want to prove that $g(x, y_\lambda, v) \nsubseteq -\mathrm{int}\, C(x)$ for all $v \in T(x)$. This will imply that $H(x)$ is convex. Suppose that there exist a λ_0 with $0 \leqslant \lambda_0 \leqslant 1$ and $v_0 \in T(x)$ such that $g(x, y_{\lambda_0}, v_0) \subseteq -\mathrm{int}\, C(x)$. By (ii), either

$$g(x, y_1, v_0) \subseteq g(x, y_{\lambda_0}, v_0) - C(x) \subseteq -\mathrm{int}\, C(x) - C(x) \subseteq -\mathrm{int}\, C(x)$$

or

$$g(x, y_2, v_0) \subseteq g(x, y_{\lambda_0}, v_0) - C(x) \subseteq -\mathrm{int}\, C(x) - C(x) \subseteq -\mathrm{int}\, C(x).$$

This is a contradiction. Therefore, $H(x)$ is convex for each $x \in X$.
H is closed. Let $(\bar{x}, \bar{y}) \in \overline{G_r H}$, then there exists a net $\{(x_\alpha, y_\alpha)\} \in G_r H$ such that $(x_\alpha, y_\alpha) \to (\bar{x}, \bar{y})$. Therefore, $y_\alpha \in T(x_\alpha)$ and $g(x_\alpha, y_\alpha, v) \nsubseteq -\mathrm{int}\, C(x_\alpha)$ for all $v \in T(x_\alpha)$. Since T is u.s.c. with closed values, it follows from Theorem 2.1 that T is closed and $\bar{y} \in T(\bar{x})$. Let $v \in T(\bar{x})$. Since T is l.s.c., it follows from Theorem 2.2 that there exists a net $\{v_\alpha\}$ such that $v_\alpha \to v$ and $v_\alpha \in T(x_\alpha)$. Since

$$g(x_\alpha, y_\alpha, v_\alpha) \nsubseteq -\mathrm{int}\, C(x_\alpha),$$
$$g(x_\alpha, y_\alpha, v_\alpha) \cap (Z \setminus (-\mathrm{int}\, C(x_\alpha)) \neq \emptyset.$$

Let $w_\alpha \in g(x_\alpha, y_\alpha, v_\alpha) \cap (Z \setminus (-\mathrm{int}\, C(x_\alpha)).$
Then $w_\alpha \in g(x_\alpha, y_\alpha, v_\alpha)$ and $w_\alpha \in Z \setminus (-\mathrm{int}\, C(x_\alpha)$.
Since $g: X \times Y \times Y \multimap Z$ is u.s.c. with compact values and $X \times Y \times Y$ is compact, $g(X \times Y \times Y)$ is compact and there exists a subnet $\{w_{\alpha_\lambda}\}$ of $\{w_\alpha\}$ such that $w_{\alpha_\lambda} \to w \in g(X \times Y \times Y)$. By (ii) and (iii), $g: X \times Y \times Y \multimap Z$ and $W: X \multimap Z$ are u.s.c. multivalued map with closed values, it follows from Theorem 2.1 that g and W are closed. $w \in g(\bar{x}, \bar{y}, v)$ and $w \in W(\bar{x}) = Z \setminus (-\mathrm{int}\, C(\bar{x}))$. Therefore, $w \in g(\bar{x}, \bar{y}, v) \cap (Z \setminus (-\mathrm{int}\, C(\bar{x}))) \neq \emptyset$. This shows that $g(\bar{x}, \bar{y}, v) \nsubseteq -\mathrm{int}\, C(\bar{x})$ for all $v \in T(\bar{x})$. From this, $(\bar{x}, \bar{y}) \in G_r H$, H is closed. Since $H: X \multimap Z$ is closed and Y is compact, it follows from Theorem 2.1 that H is an u.s.c. multivalued map with closed convex values. We

want to show that $H(x)$ is nonempty for all $x \in X$. For each fixed $x \in X$, let $G_x: T(x) \multimap T(x)$ be defined by $G_x(v) = \{y \in T(x): g(x, y, v) \nsubseteq -\mathrm{int}\, C(x)\}$. Suppose that there exists a finite set $\{v_1, v_2, \ldots, v_n\}$ in $T(x)$ such that $co\{v_1, v_2, \ldots, v_n\} \nsubseteq \bigcup_{i=1}^{n} G_x(v_i)$. Then there exists $\bar{y} \in co\{v_1, v_2, \ldots, v_n\}$ such that $\bar{y} \notin G_x(v_i)$ for all $i = 1, 2, \ldots, n$. Since $T(x)$ is a convex for all $x \in X$, $\bar{y} \in T(x)$ and $g(x, \bar{y}, v_i) \subseteq -\mathrm{int}\, C(x)$ for all $i = 1, 2, \ldots, n$. By (ii) and Proposition 2.6 that there exists $1 \leqslant j \leqslant n$ such that

$$g(x, \bar{y}, v_j) \subseteq g(x, \bar{y}, \bar{y}) + C(x) \subseteq C(x) + C(x) \subseteq C(x).$$

By assumption, $C(x)$ is a proper cone in Z, $C(x) \cap (-\mathrm{int}\, C(x)) = \emptyset$, and $g(x, \bar{y}, v_j) \cap (-\mathrm{int}\, C(x)) = \emptyset$. This contradicts with $g(x, \bar{y}, v_i) \subseteq -\mathrm{int}\, C(x)$ for all $i = 1, 2, \ldots, n$. Therefore, G_x is a KKM map.

Write the same argument as before, we show $G_x(v)$ is closed for each $x \in X$ and $v \in Y$, since $G_x(v) \subseteq T(x)$ and $T(x)$ is a compact set, $G_x(v)$ is a compact set for each $v \in X$. By Theorem 2.7, $\bigcap_{v \in T(x)} G_x(v) \neq \emptyset$. Let $y \in \bigcap_{v \in T(x)} G_x(v)$. Then $y \in T(x)$ and $g(x, y, v) \nsubseteq -\mathrm{int}\, C(x)$ for all $v \in T(x)$. Therefore, $y \in H(x) \neq \emptyset$ for each $x \in X$. By (iv)(b), it is easy to see that for all $(x, y) \in X \times Y$, $x \notin \mathrm{co} P(x, y)$. By (iv)$(a)$ $P^{-}(u)$ is open for all $u \in X$. Then it follows from Theorem 2.3 that there exists $(\bar{x}, \bar{y}) \in X \times Y$, $\bar{x} \in cl S(\bar{x})$ and $\bar{y} \in H(\bar{x})$ such that $S(\bar{x}) \cap P(\bar{x}, \bar{y}) = \emptyset$. Therefore, $\bar{y} \in T(\bar{x})$, $f(\bar{x}, \bar{y}, u) \nsubseteq -\mathrm{int}\, C(\bar{x})$ for all $u \in S(\bar{x})$ and $g(\bar{x}, \bar{y}, v) \nsubseteq -\mathrm{int}\, C(\bar{x})$ for all $v \in T(\bar{x})$.

REMARK. If $g = 0$, then Theorem 3.2 is reduced to Theorem 3.1 in Hou et al. (2003).

COROLLARY 3.2. *In Theorem* 3.2, *if condition* (iv) *(a) is replaced by (a′) for each fixed $u \in X$, $(x, y) \multimap f(x, y, u)$ is u.s.c. with compact values. Then the conclusion of Theorem* 3.2 *holds.*

Proof. Following the same arguments as in Theorem 3.2, we can show that for each $u \in X$, $A(u) = \{(x, y) \in X \times Y: f(x, y, u) \cap (Z \backslash (-\mathrm{int}\, C(x)) \neq \emptyset\}$ is closed.

THEOREM 3.3. *Let X, Y, Z, S and T be the same as in Theorem 3.1. Suppose that*

(i) *$C: X \multimap Z$ is a closed multivalued map with nonempty convex values;*
(ii) *g is closed and $g(x, y, y) \subseteq C(x)$ for all $x \in X$ and $y \in Y$;*
(iii) *for each $(x, v) \in X \times Y$, $y \multimap g(x, y, v)$ is concave, and g is compact, and for each $(x, y) \in X \times Y$, $v \multimap g(x, y, v)$ is C_x-quasiconvex;*
(iv) *(a) for all $u \in X$, the set*

$\{(x,y)\in X\times Y: f(x,y,u)\cap C(x)=\emptyset\}$ *is open in* $X\times Y$; *and*

(b) for all $y\in Y$, $f(\cdot,y,\cdot)$ *is SIIC-DQC in the third argument.*

Then there exists $(\bar{x},\bar{y})\in X\times Y$, $\bar{x}\in clS(\bar{x})$ and $\bar{y}\in T(\bar{x})$ such that

$$g(\bar{x},\bar{y},v)\cap C(\bar{x})\neq\emptyset \quad \text{for all } v\in T(\bar{x})$$

and

$$f(\bar{x},\bar{y},u)\cap C(\bar{x})\neq\emptyset \quad \text{for all } u\in S(\bar{x}).$$

Proof. Let $H: X \multimap Y$ and $P: X\times Y \multimap$ be defined by

$$H(x)=\{y\in T(x): g(x,y,v)\cap C(x)\neq\emptyset \quad \text{for all } v\in T(x)\},$$

$$P(x,y)=\{u\in X: f(x,y,u)\cap C(x)=\emptyset\} \quad \text{for}(x,y)\in X\times Y.$$

Following similar arguments as in Theorems 3.1 and 3.2, we can show that $H(x)$ is nonempty for all $x\in X$.

Let $y_1,y_2\in H(x)$, $\lambda\in[0,1]$. Then $y_1,y_2\in T(x)$, $g(x,y_1,v)\cap C(x)\neq\emptyset$ and $g(x,y_2,v)\cap C(x)\neq\emptyset$ for all $v\in T(x)$. Let $z_1\in g(x,y_1,v)\cap C(x)$ and $z_2\in g(x,y_2,v)\cap C(x)$. Then $\lambda z_1+(1-\lambda)z_2\in C(x)$. For each fixed $x\in X$, and $v\in T(x)$, by (iii), $\lambda z_1+(1-\lambda)z_2\in\lambda g(x,y_1,v)+(1-\lambda)g(x,y_2,v)\subseteq g(x,\lambda y_1+(1-\lambda)y_2,v)$. Therefore, $\lambda z_1+(1-\lambda)z_2\in g(x,\lambda y_1+(1-\lambda)y_2,v)\cap C(x)\neq\emptyset$ for all $v\in T(x)$. Since $T(x)$ is convex, $\lambda y_1+(1-\lambda)y_2\in T(x)$ This shows that $H(x)$ is convex. With the same argument as in Theorems 3.1 and 3.2, we can show that H is closed. Since $H: X\multimap Y$ is closed and Y is compact. H is an u.s.c. multivalued map with nonempty closed convex valued. By (iv)(*b*), it is easy to see that for all $(x,y)\in X\times Y$, $x\notin \mathrm{co}P(x,y)$. By (iv)(*a*), $P^{-}(u)$ is open for all $u\in X$. Then it follows from Theorem 2.3 that there exists $(\bar{x},\bar{y})\in X\times Y$, $\bar{x}\in S(\bar{x})$, $\bar{y}\in H(\bar{x})$ such that $S(\bar{x})\cap P(\bar{x},\bar{y})=\emptyset$. Therefore, $\bar{y}\in T(\bar{x})$, $f(\bar{x},\bar{y},u)\cap C(\bar{x})\neq\emptyset$ for all $u\in S(\bar{x})$, and $g(\bar{x},\bar{y},v)\cap C(\bar{x})\neq\emptyset$ for all $v\in T(\bar{x})$.

REMARK. (1) Condition v(*a*) of Theorem 3.3 is satisfied if either C is closed and f is u.s.c. with compact values or f is closed and C is u.s.c. with compact values.

(2) If $g=0$, then Theorem 3.3 is reduced to Theorem 3.4 in Hou et al. (2003).

THEOREM 3.4. *Let X a nonempty closed convex metrizable sets in locally convex t.v.s. and Y be a nonempty complete convex metrizable sets in locally* *convex t.v.s., Z be a real t.v.s. and $S: X \multimap X$ be a multivalued map. Let C, g, f be the same as in Theorem* 3.1. *Suppose further that $A \subseteq X$ is a nonempty compact convex subsets and $B \subseteq A$ is a nonempty subset such that*

(i) *$T: X \multimap Y$ is a continuous multivalued map with nonempty compact convex values and;*
(ii) *$S(B) \subseteq A$;*
(iii) *for each $x \in X$, the set $S_0(x) = S(x) \cap A$ is a nonempty convex set, $S_0^-(y)$ is open for all $y \in A$ and clS_0 is an u.s.c. map;*
(iv) *For each $x \in A \backslash B$ satisfying $x \in clS_0(x)$ and for each $y \in T(x)$ either there exists $u_x \in S_0(x)$ such that $f(x, y, u_x) \nsubseteq C(x)$ or there exists $v_x \in T(x)$ such that $g(x, y, v_x) \nsubseteq C(x)$.*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$, $\bar{x} \in clS(\bar{x})$ and $\bar{y} \in T(\bar{x})$ such that

$$f(\bar{x}, \bar{y}, u) \subseteq C(\bar{x}) \quad \text{for all } u \in clS(\bar{x})$$

and

$$g(\bar{x}, \bar{y}, v) \subseteq C(\bar{x}) \quad \text{for all } v \in T(\bar{x}).$$

Proof. Let $Y_0 = \overline{co} T(A)$. Since $T: X \multimap Y$ is an u.s.c. multivalued map with nonempty compact values and A is compact, it follows from Theorems 2.1 and 2.5 that $T(A)$ is compact and Y_0 is a compact convex subset of Y. By Theorem 3.1 that there exists $(\bar{x}, \bar{y}) \in A \times Y_0$, $\bar{x} \in S_0(\bar{x})$ and $\bar{y} \in T(\bar{x})$ such that

$$f(\bar{x}, \bar{y}, u) \subseteq C(\bar{x}) \quad \text{for all } u \in S_0(\bar{x})$$

and

$$g(\bar{x}, \bar{y}, v) \subseteq C(\bar{x}) \quad \text{for all } v \in T(\bar{x}).$$

If $\bar{x} \in A \backslash B$, by (iv), there exists $u_x \in S_0(\bar{x})$ such that $f(\bar{x}, \bar{y}, u_x) \nsubseteq C(\bar{x})$ or there exists $v_x \in T(\bar{x})$ such that $g(\bar{x}, \bar{y}, v_x) \nsubseteq C(\bar{x})$. This leads to a contradiction. Therefore, $\bar{x} \in B$, $S_0(\bar{x}) = S(\bar{x}) \cap A = S(\bar{x})$ and Theorem 3.4 follows.

THEOREM 3.5. *Let Z, X, Y and T be the same as Theorem* 3.1 *Suppose that*

(i) *$W: X \multimap Z$ defined by $W(x) = Z \backslash -\operatorname{int} C(x)$ is u.s.c., $C(x)$ is a cone and $W(x)$ is a convex set for all $x \in X$;*
(ii) *g is l.s.c., and $g(x, y, y) \subseteq C(x)$ for all $(x, y) \in X \times Y$;*
(iii) *for each fixed $(x, v) \in X \times Y$, $y \multimap g(x, y, v)$ is convex, and for each fixed $(x, y) \in X \times Y$, $v \multimap g(x, y, v)$ is C_x-quasiconvex;*

(iv) *(a) for any* $u \in X$, *the set*

$$\{(x, y) \in X \times Y : f(x, y, u) \cap (-\operatorname{int} C(x)) \neq \emptyset\} \text{ is open in } X \times Y.$$

(b) for any $y \in Y$, *the function* $f(\cdot, y, \cdot)$ *is WIC-DQC in the third argument.*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$, $\bar{x} \in clS(\bar{x})$ and $\bar{y} \in T(\bar{x})$ such that

$$f(\bar{x}, \bar{y}, u) \cap (-\operatorname{int} C(\bar{x})) = \emptyset \quad \text{for all } u \in S(\bar{x})$$

and

$$g(\bar{x}, \bar{y}, v) \cap (-\operatorname{int} C(\bar{x})) = \emptyset \quad \text{for all } v \in T(\bar{x}).$$

Proof. Let $H: X \multimap Y$ and $P: X \times Y \multimap X$ be defined by

$$H(x) = \{y \in T(x) : g(x, y, v) \cap (-\operatorname{int} C(x)) = \emptyset \quad \text{for all } v \in T(x)\} \quad \text{and,}$$
$$P(x, y) = \{u \in X : f(x, y, u) \cap (-\operatorname{int} C(x)) \neq \emptyset\} \quad \text{for}(x, y) \in X \times Y.$$

Then following the similar arguments as in Theorems 3.1 and 3.2, we can show that $H(x) \neq \emptyset$ for all $x \in X$ and H is closed. Since $H: X \multimap Y$ is closed and Y is compact, H is an u.s.c. multivalued map with closed values. Let $y_1, y_2 \in H(x)$ and $\lambda \in [0, 1]$, then $y_1, y_2 \in T(x)$, $g(x, y_1, v) \subseteq W(x)$ and $g(x, y_2, v) \subseteq W(x)$ for all $v \in T(x)$. Since $T(x)$ and $W(x)$ are convex sets for all $x \in X$ and g is convex,

$$g(x, \lambda y_1 + (1 - \lambda) y_2, v) \subseteq \alpha g(x, y_1, v) + (1 - \alpha) g(x, y_2, v) \subseteq W(x)$$
$$\text{for all } v \in T(x)$$

and $\lambda y_1 + (1 - \lambda) y_2 \in T(x)$. Therefore, $H(x)$ is a convex set for each $x \in X$. By (iv)(*b*), it is easy to show that for all $(x, y) \in X \times Y$, $x \notin \operatorname{co}P(x, y)$, By (iv)(*a*) $P^-(u)$ is open for all $u \in X$. Then it follows from Theorem 2.3 that there exists $(\bar{x}, \bar{y}) \in X \times Y$, $\bar{x} \in clS(\bar{x})$ and $\bar{y} \in H(\bar{x})$ such that $S(\bar{x}) \cap P(\bar{x}, \bar{y}) = \emptyset$. Therefore,

$$f(\bar{x}, \bar{y}, u) \cap (-\operatorname{int} C(\bar{x})) = \emptyset \quad \text{for all } u \in S(\bar{x}).$$

From this, $\bar{y} \in T(\bar{x})$ and

$$g(\bar{x}, \bar{y}, v) \cap (-\operatorname{int} C(\bar{x})) = \emptyset \quad \text{for all } v \in T(\bar{x}).$$

REMARK. (1) Condition (iv)(*a*) is satisfied if for each $u \in X$, $(x, y) \multimap f(x, y, u)$ is l.s.c. and $W: X \multimap Z$ is u.s.c., where $W(x) = Z \backslash (-\operatorname{int} C(x))$.

(2) If $g = 0$, then Theorem 3.5 is reduced to Theorem 3.3 in Hou et al. (2003).

4. Applications in optimization theory

Applying theorem 3.1, we have the following existence theorems of simultaneous optimization problems and equilibrium problems. We also establish the existence theorems of generalized vector quasi-saddle point problems.

THEOREM 4.1. *Let* X, Y *and* Z *be the same as in Theorem* 3.1. *Suppose that*

(i) $S: X \multimap X$ *is a multivalued map with nonempty convex values and* $S^{-}(y)$ *is open for all* $y \in X$ *and* $clS: X \multimap X$ *is u.s.c.;*
(ii) $T: X \multimap Y$ *is a continuous multivalued map with nonempty closed convex values;*
(iii) $C: X \multimap Z$ *is a closed multivalued map such that for each* $x \in X$, $C(x)$ *is a nonempty convex cone;*
(iv) $g: X \times Y \times Y \multimap Z$ *is a l.s.c. multivalued map, and for each* $(x, y) \in X \times Y$, $g(x, y, y) \subset C(x)$*;*
(v) *for each fixed* $(x, v) \in X \times Y$, $y \multimap g(x, y, v)$ *is* C_x*-quasiconcave, and for each fixed* $(x, y) \in X \times Y$, $v \multimap g(x, y, v)$ *is* C_x*-quasiconvex like;*

  (a) $f: X \to Z$ *is a continuous function;*
  (b) $h: X \times X \to Z$, $h(x, u) = f(u) - f(x)$ *is strong type* I C*-diagonally quasi-convex in the second argument.*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ with $\bar{x} \in clS(\bar{x})$ and $\bar{y} \in T(\bar{x})$ such that

$$f(u) - f(\bar{x}) \in C(\bar{x}) \quad \text{for all } u \in S(\bar{x})$$

and

$$g(\bar{x}, \bar{y}, v) \subseteq C(\bar{x}) \qquad \text{for all } v \in T(\bar{x}).$$

Proof. Let $F(x, y, u) = f(u) - f(x)$. Since f is continuous and C is closed, it is easy to see that for any $u \in X$, the set $\{(x, y) \in X \times Y,\ f(u) - f(x) \in C(x)\}$ is closed.

Therefore, the conclusion of Theorem 4.1 follows from Theorem 3.1. □

THEOREM 4.2. *Let* X, Y, Z, S, T *and* C *be the same as as Theorem* 4.1. *Let* $f: X \times Y \to Z$ *be a continuous function satisfying the following conditions:*

(i) *For each fixed* $x \in X$, $y \to f(x, y)$ *is* C_x*-quasiconcave;*
(ii) *For each fixed* $y \in Y$ *and any finite set* $A = \{x_1, \ldots, x_n\}$ *in* X, *and any* $x \in \mathrm{co}\{x_1, x_2, \ldots, x_n\}$, $f(x_i, y) - f(x, y) \in C(x)$ *for some* i.

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ with $\bar{x} \in clS(\bar{x})$ and $\bar{y} \in T(\bar{x})$ such that

$$f(x, \bar{y}) - f(\bar{x}, \bar{y}) \in C(\bar{x}) \quad \text{for all } x \in S(\bar{x})$$

and

$$f(\bar{x}, \bar{y}) - f(\bar{x}, y) \in C(\bar{x}) \quad \text{for all } y \in T(\bar{x}).$$

Proof. Let $F(x, y, u) = f(u, y) - f(x, y)$ and

$$g(x, y, v) = f(x, y) - f(x, v),$$

then for each fixed $(x, y) \in X \times Y$, $g(x, y, y) = 0 \in C(x)$. Since for each fixed $x \in X$, $y \to f(x, y)$ is C_x-quasiconcave, for each fixed (x, v), $y \to g(x, y, v)$ is C_x-quasiconcave, and for each fixed $(x, y) \in X \times Y$, $v \to g(x, y, v)$ is C_x-quasiconconvex. By (ii), for any $y \in Y$, $F(\cdot, y, \cdot)$ is strong type I C-diagonally quasi-convex in the third argument. Since $f : X \times Y \multimap Z$ is continuous, F and g are continuous. Since C is closed, for any $u \in X$, the set

$$\{(x, y) \in X \times Y : F(x, y, u) \notin C(x)\} \quad \text{is open in } X \times Y.$$

Then it follows from Theorem 3.1 that there exists $(\bar{x}, \bar{y}) \in X \times Y$, $\bar{x} \in clS(\bar{x})$ and $\bar{y} \in T(\bar{x})$ such that

$$F(\bar{x}, \bar{y}, u) \in C(\bar{x}) \quad \text{for all } u \in S(\bar{x})$$

and

$$g(\bar{x}, \bar{y}, v) \in C(\bar{x}) \quad \text{for all } v \in T(\bar{x}).$$

From this,

$$f(x, \bar{y}) - f(\bar{x}, \bar{y}) \in C(\bar{x}) \quad \text{for all } x \in S(\bar{x})$$

and

$$f(\bar{x}, \bar{y}) - f(\bar{x}, v) \in C(\bar{x}) \quad \text{for all } v \in T(\bar{x}).$$

When X and Y are not compact, we applying Theorem 4.2 and following the same arguments as in Theorem 3.4, we have the following theorem.

THEOREM 4.3. *Let* X, Y, Z, S, T, C, S_0, A *and* B *be the same as Theorem* 3.4 *satisfying conditions* (i)–(iii) *of Theorem* 3.4*. Let* $f : X \times Y \to Z$ *be a continuous function satisfying the following conditions:*

(i) *for each fixed* $x \in X$, $y \to f(x, y)$ *is* C_x*-quasiconcave;*
(ii) *for each fixed* $y \in Y$ *and any finite set* $M = \{x_1, \ldots, x_n\}$ *in* X *and any* $x \in \text{co}\{x_1, x_2, \ldots, x_n\}$, $f(x_i, y) - f(x, y) \in C(x)$ *for some* i*;*
(iii) *for each* $x \in A \setminus B$ *satisfying* $x \in cl\, S_0(x)$ *and for each* $y \in T(x)$, *either there exists* $u_x \in S_0(x)$ *such that* $f(u_x, y) - f(x, y) \notin C(x)$ *or there exists* $v_x \in T(x)$ *such that* $f(x, y) - f(x, v_x) \notin C(x)$.

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$, $\bar{x} \in cl\, S(\bar{x})$ and $\bar{y} \in T(\bar{x})$ such that

$$f(x, \bar{y}) - f(\bar{x}, \bar{y}) \in C(\bar{x}) \quad \text{for all } x \in S(\bar{x})$$

and

$$f(\bar{x}, \bar{y}) - f(\bar{x}, y) \in C(\bar{x}) \quad \text{for all } y \in T(\bar{x}).$$

THEOREM 4.4. *Let* Z *be a real Banach space,* X *be a compact Banach space.* Y *be a nonempty compact convex metrizable subset of a locally convex t.v.s. and* $L(X, Z)$ *be equipped with* $\sigma-$*topology. Suppose that*

(i) $S: X \multimap X$ *is a multivalued map with nonempty convex values,* $S^-(y)$ *is open for all* $y \in Y$ *and* $clS: X \multimap X$ *is an u.s.c. multivalued map;*
(ii) $T: X \multimap Y$ *is an u.s.c. multivalued map with nonempty compact convex values;*
(iii) $C: X \multimap Z$ *is a closed multivalued map such that for each* $x \in X$, $C(x)$ *is a nonempty convex cone;*
(iv) *(a) the function* $(x, y) \to f'_x(x, y)$ *is continuous on* $X \times Y$, *where* $f'_x(x, y)$ *is the Fréchet derivative of* f *at* (x, y) *with respect to* x*;*
(b) for any $y \in Y$, *and any finite subset* $A = \{x_1, x_2, \ldots, x_n\}$ *in* X *and any* $x \in \text{co}\{x_1, x_2, \ldots, x_n\}$, $\langle f'_x(x, y), x_i - x \rangle \in C(x)$ *for some* i*;*
(v) $f: X \times Y \to Z$ *is a continuous function and for each fixed* $x \in X$, $y \to f(x, y)$ *is* C_x*-quasiconcave.*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$, $\bar{x} \in cl\, S(\bar{x})$ and $\bar{y} \in T(\bar{x})$ such that

$$\langle f'_x(\bar{x}, \bar{y}), x - \bar{x} \rangle \in C(\bar{x}) \quad \text{for all } x \in S(\bar{x})$$

and

$$f(\bar{x}, \bar{y}) - f(\bar{x}, y) \in C(\bar{x}) \quad \text{for all } y \in T(\bar{x}).$$

Proof. Let $F: X \times Y \times X \to Z$ and $g: X \times Y \times Y \to Z$ be defined by $F(x, y, u) = \langle f'_x(x, y), u - x \rangle$ and $g(x, y, v) = f(x, y) - f(x, v)$. Then by (iv), g is continuous, by (iv), for each $(x, v) \in X \times Y$, $y \to g(x, y, v)$ is C_x-quasiconcave, and for each $(x, y) \in X \times Y$, $v \to g(x, y, v)$ is C_x-quasiconconvex and $g(x, y, y) = 0 \in C(x)$.

By (iv)(a) and Theorem 2.4, for each $u \in X$, $(x, y) \to < f'(x, y), u - x >$ is continuous. Since C is closed, it is easy to see that for each $u \in X$, $\{(x, y) \in X \times Y : \langle f_x'(x, y), u - x\rangle \in C(x)\}$ is closed. Then the conclusion of Theorem 4.4 follows from Theorem 3.1.

THEOREM 4.5. *Under the assumption of Theorem* 4.4. *We further assume that for each* $y \in Y$, $x \to f(x, y)$ *is* $C(x)$*-convex. Then there exists* $(\bar{x}, \bar{y}) \in X \times Y$, $\bar{x} \in cl S(\bar{x})$ *and* $\bar{y} \in T(\bar{x})$ *such that*

$$f(x, \bar{y}) - f(\bar{x}, \bar{y}) \in C(\bar{x}) \quad \textit{for all } x \in S(\bar{x})$$

and

$$f(\bar{x}, y) - f(\bar{x}, v) \in C(\bar{x}) \quad \textit{for all } y \in T(\bar{x}).$$

Proof. It follows from Theorem 4.4 that there exists $(\bar{x}, \bar{y}) \in X \times Y$, $\bar{x} \in cl S(\bar{x})$ and $\bar{y} \in T(\bar{x})$ such that

$$\langle f_x'(\bar{x}, \bar{y}), x - \bar{x}\rangle \in C(\bar{x}) \quad \text{for all } x \in S(\bar{x})$$

and

$$f(\bar{x}, \bar{y}) - f(\bar{x}, y) \in C(\bar{x}) \quad \text{for all } y \in T(\bar{x}).$$

Since for each $y \in Y$, $x \to f(x, y)$ is $C(x)$ convex, for each $x \in X$ and $\alpha \in (0, 1)$

$$-f(\bar{x} + \alpha(x - \bar{x}), \bar{y}) + \alpha f(x, \bar{y}) + (1 - \alpha) f(\bar{x}, \bar{y}) \in C(\bar{x} + \alpha(x - \bar{x})),$$
$$f(x, \bar{y}) - f(\bar{x}, \bar{y}) - \frac{f(\bar{x} + \alpha(x - \bar{x}), \bar{y}) - f(\bar{x}, \bar{y})}{\alpha} \in C(\bar{x} + \alpha(x - \bar{x})).$$

Since $C: X \multimap Z$ is a closed multivalued map, and

$$\lim_{\alpha \to 0} \frac{f(\bar{x} + \alpha(x - \bar{x}), \bar{y}) - f(\bar{x}, \bar{y})}{\alpha} = < f_x'(\bar{x}, \bar{y}), x - \bar{x} >,$$

$$\begin{aligned} f(x, \bar{y}) - f(\bar{x}, \bar{y}) &\in \langle f_x'(\bar{x}, \bar{y}), x - \bar{x})\rangle + C(\bar{x}) \\ &\subseteq C(\bar{x}) + C(\bar{x}) \subseteq C(\bar{x}). \end{aligned}$$

This completes the proof of Theorem 4.5. □

If X and Y are not compact, then we have the following theorem.

THEOREM 4.6. *Let Z be a real Banach space, X be a Banach space, Y be a complete metrizable locally convex t.v.s. and $L(E,Z)$ be equipped with σ-topology. Suppose that $S: X \multimap X$ is a multivalued map, $T: X \multimap Y$ is an u.s.c. multivalued map with nonempty compact convex values, $C: X \multimap Z$ is a closed multivalued map such that for each $x \in X$, $C(x)$ is a nonempty convex cone. Suppose further that $A \subseteq X$ is a nonempty compact convex subsets and $B \subseteq A$ is a nonempty subset and $f: X \times Y \to Z$ is a continuous function satisfying the following conditions:*

(i) *for each fixed $x \in X$, $y \to f(x,y)$ is C_x-quasiconcave, for each fixed $y \in Y$, $x \to f(x,y)$ is $C(x)$-convex and the function $(x,y) \to f'_x(x,y)$ is continuous on $X \times Y$, where $f'_x(x,y)$ is the Fréchet derivative of f at (x,y) with respect to x;*
(ii) *for any fixed $y \in Y$ and any finite subset $M = \{x_1, x_2, \ldots, x_n\}$ in X and any $x \in \mathrm{co}\{x_1, x_2, \ldots, x_n\}$, $\langle f'_x(x,y), x_i - x \rangle \in C(x)$ for some i;*
(iii) *$S(B) \subseteq A$;*
(iv) *for each $x \in X$, $S_0(x) = S(x) \cap A$ is a nonempty convex set, $S_0^-(y)$ is open for all $y \in A$ and $\mathrm{cl}S_0 : X \multimap X$ is an u.s.c. multivalued map;*
(v) *for each $x \in A \backslash B$, satisfying $x \in S_0(x)$ and for each $y \in T(x)$ either there exists $u_x \in S_0(x)$ such that*

$$f(u_x, y) - f(x,y) \notin C(x)$$

or there exists $v_x \in T(x)$ such that

$$f(x,y) - f(x, v_x) \notin C(x).$$

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$, $\bar{x} \in clS(\bar{x})$ and $\bar{y} \in T(\bar{x})$ such that

$$f(x,\bar{y}) - f(\bar{x},\bar{y}) \in C(\bar{x}) \quad \text{for all } x \in S(\bar{x})$$

and

$$f(\bar{x},\bar{y}) - f(\bar{x},y) \in C(\bar{x}) \quad \text{for all } y \in T(\bar{x}).$$

Proof. Let $Y_0 = \overline{co}T(A)$. Then Y_0 is a compact convex set in Y. By Theorem 4.5 that there exists $(\bar{x},\bar{y}) \in A \times Y_0$, $\bar{x} \in clS_0(\bar{x})$ and $\bar{y} \in T(\bar{y})$ such that

$$f(x,\bar{y}) - f(\bar{x},\bar{y}) \in C(\bar{x}) \quad \text{for all } x \in S_0(\bar{x})$$

and

$$f(\bar{x},\bar{y}) - f(\bar{x},y) \in C(\bar{x}) \quad \text{for all } y \in T(\bar{x}).$$

Following the same arguments as in Theorem 3.4, we prove that $\bar{x} \in B$, $S_0(\bar{x}) = S(\bar{x})$ and Theorem 4.6 follows.

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